Dualities and representations of Hecke algebras for interacting particle systems

Roger Tribe, Bruce Westbury and Oleg Zaboronski

Department of Mathematics, University of Warwick

July 09, 2020

Outline

How it all started

- Idea (**Smoluchowski, 1907**): use Boltzmann theory to study chemical kinetics

How it all started

- Idea (**Smoluchowski, 1907**): use Boltzmann theory to study chemical kinetics

- The simplest models:
  1. $A + A \to \emptyset$ (annihilation)
  2. $A + A \to A$ (coalescence)

How it all started

- Idea (**Smoluchowski, 1907**): use Boltzmann theory to study chemical kinetics
- The simplest models:
  1. $A + A \to \emptyset$ (annihilation)
  2. $A + A \to A$ (coalescence)
- Experimental realisation (**2000's**): localised excitations in nanowires

Mean field analysis

- ρ_t - density of reactants

Mean field analysis

- ρ_t - density of reactants
- Rate equation:

$$\dot{\rho}_t = - \underbrace{\lambda}_{Reaction\ \ rate} \times \underbrace{\frac{1}{2}\rho_t^2}_{Number\ \ of\ \ reacting\ \ pairs\ \ in\ \ unit\ \ volume}$$

Mean field analysis

- ρ_t - density of reactants
- Rate equation:

$$\dot{\rho}_t = -\underbrace{\lambda}_{Reaction\ \ rate} \times \underbrace{\frac{1}{2}\rho_t^2}_{Number\ \ of\ \ reacting\ \ pairs\ \ in\ \ unit\ \ volume}$$

- Large-t asymptotic: $\rho_t = \frac{2}{\lambda t}(1 + O(t^{-1}))$

Mean field analysis

- ρ_t - density of reactants
- Rate equation:

$$\dot{\rho}_t = -\underbrace{\lambda}_{Reaction\ rate} \times \underbrace{\frac{1}{2}\rho_t^2}_{Number\ of\ reacting\ pairs\ in\ unit\ volume}$$

- Large-t asymptotic: $\rho_t = \frac{2}{\lambda t}(1 + O(t^{-1}))$
- **Implicit mean field assumption**: absence of spatial correlations

Mean field analysis

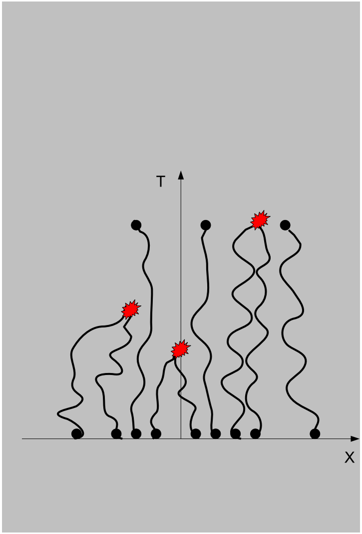
- ρ_t - density of reactants
- Rate equation:

$$\dot{\rho}_t = -\underbrace{\lambda}_{\textit{Reaction rate}} \times \underbrace{\frac{1}{2}\rho_t^2}_{\textit{Number of reacting pairs in unit volume}}$$

- Large-t asymptotic: $\rho_t = \frac{2}{\lambda t}(1 + O(t^{-1}))$
- **Implicit mean field assumption**: absence of spatial correlations
- So, the probability of finding particles at n disjoint positions is

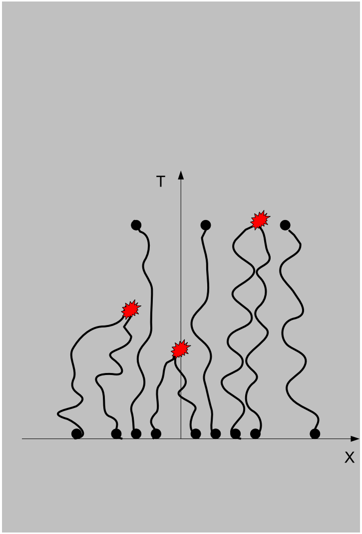
$$\rho_t^n \overset{t\to\infty}{\sim} t^{-n}$$

Annihilating Random Walks



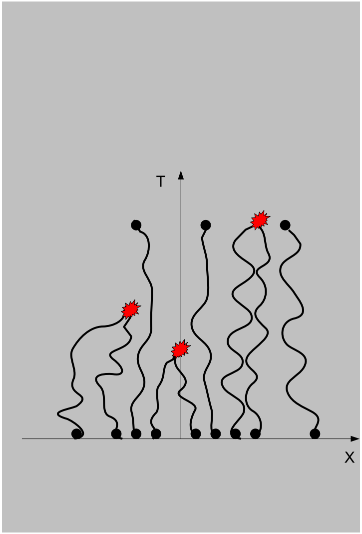
- Particles perform independent CT RW's on $\mathbb{Z}$ until they meet

Annihilating Random Walks



- Particles perform independent CT RW's on $\mathbb{Z}$ until they meet
- At the moment of collision particles instantly annihilate.

Annihilating Random Walks

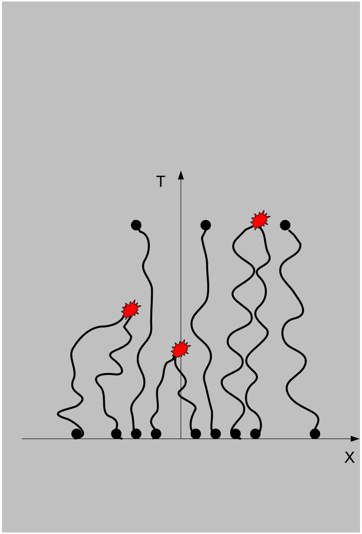


- Particles perform independent CT RW's on $\mathbb{Z}$ until they meet
- At the moment of collision particles instantly annihilate.
- **Correlation functions**:

$$\rho_t^{(n)}(x_1, \ldots, x_n)$$

- the probability of finding particles at $x_1, \ldots, x_n$ at time t

Annihilating Random Walks



- Particles perform independent CT RW's on $\mathbb{Z}$ until they meet
- At the moment of collision particles instantly annihilate.
- **Correlation functions**:

$$\rho_t^{(n)}(x_1, \ldots, x_n)$$

  - the probability of finding particles at $x_1, \ldots, x_n$ at time t

- Continuous limit for $d = 1$ ($x_c = \epsilon x$, $t_c = \epsilon^2 t$): annihilating BM's on $\mathbb{R}$

A bit of history

- **Contributors**: Smoluchowski, Glauber, Bramson, Lebowitz, Griffeath, Doi, Zeldovich, Ovchinnikov, Peliti, Droz, Lee, Cardy, Kesten, Derrida, Zeitak, Hakim, Pasquier, ben Avraham, Masser, Ben-Naim, Krapivsky, Connaughton, R. Rajesh, Warren, R. Sun...

A bit of history

- **Contributors**: Smoluchowski, Glauber, Bramson, Lebowitz, Griffeath, Doi, Zeldovich, Ovchinnikov, Peliti, Droz, Lee, Cardy, Kesten, Derrida, Zeitak, Hakim, Pasquier, ben Avraham, Masser, Ben-Naim, Krapivsky, Connaughton, R. Rajesh, Warren, R. Sun...
- **Connection to spin chains for** $d = 1$: domain walls in the Glauber model at $T = 0$

A bit of history

- **Contributors**: Smoluchowski, Glauber, Bramson, Lebowitz, Griffeath, Doi, Zeldovich, Ovchinnikov, Peliti, Droz, Lee, Cardy, Kesten, Derrida, Zeitak, Hakim, Pasquier, ben Avraham, Masser, Ben-Naim, Krapivsky, Connaughton, R. Rajesh, Warren, R. Sun. . .
- **Connection to spin chains for** $d = 1$: domain walls in the Glauber model at $T = 0$

Results:

A bit of history

- **Contributors**: Smoluchowski, Glauber, Bramson, Lebowitz, Griffeath, Doi, Zeldovich, Ovchinnikov, Peliti, Droz, Lee, Cardy, Kesten, Derrida, Zeitak, Hakim, Pasquier, ben Avraham, Masser, Ben-Naim, Krapivsky, Connaughton, R. Rajesh, Warren, R. Sun...
- **Connection to spin chains for $d = 1$**: domain walls in the Glauber model at $T = 0$

Results:

- Mean field: $\rho_t^{(1)} \sim t^{-1}$ (1907)
- $d = 1$: $\rho_t^{(1)} \sim t^{-1/2}$ (1980's)
- $d = 1$: $\rho_t^{(n)} \sim t^{-\frac{n}{2}-\frac{n(n-1)}{4}}$ (2011)
- $d > 2$: $\rho_t^{(1)} \sim t^{-1}$ (1990's)
- $d = 2$: $\rho_t^{(1)} \sim \frac{\log t}{t}$ (1980's)
- $d = 2$: $\rho_t^{(n)} \sim \frac{(\log t)^{n-\frac{n(n-1)}{2}}}{t^n}$ (2018)

ARW's on $\mathbb{Z}$ defined

- **State space**: $\Omega = \{0,1\}^{\mathbb{Z}}$, $\eta \in \Omega$- a configuration

ARW's on $\mathbb{Z}$ defined

- **State space**: $\Omega = \{0,1\}^{\mathbb{Z}}$, $\eta \in \Omega$- a configuration
- $\eta(x) = 1$ - a particle at site $x \in \mathbb{Z}$; $\eta(x) = 0$ - site x is empty

ARW's on $\mathbb{Z}$ defined

- **State space**: $\Omega = \{0,1\}^{\mathbb{Z}}$, $\eta \in \Omega$- a configuration
- $\eta(x) = 1$ - a particle at site $x \in \mathbb{Z}$; $\eta(x) = 0$ - site x is empty
- **Model generator**:
  - $\partial_t \mathbb{E}\left[F(\eta_t)\right] = \mathbb{E}\left[LF(\eta_t)\right]$

ARW's on $\mathbb{Z}$ defined

- **State space**: $\Omega = \{0,1\}^{\mathbb{Z}}$, $\eta \in \Omega$- a configuration
- $\eta(x) = 1$ - a particle at site $x \in \mathbb{Z}$; $\eta(x) = 0$ - site x is empty
- **Model generator**:
  - $\partial_t \mathbb{E}[F(\eta_t)] = \mathbb{E}[LF(\eta_t)]$
  - $L = \sum_{x\in\mathbb{Z}}(\underbrace{\sigma_i}_{acts\ on\ \eta_i,\eta_{i+1}} - I)$, $\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

ARW's on $\mathbb{Z}$ defined

- **State space**: $\Omega = \{0,1\}^{\mathbb{Z}}$, $\eta \in \Omega$- a configuration
- $\eta(x) = 1$ - a particle at site $x \in \mathbb{Z}$; $\eta(x) = 0$ - site x is empty
- **Model generator**:
  - $\partial_t \mathbb{E}\left[F(\eta_t)\right] = \mathbb{E}\left[LF(\eta_t)\right]$
  - $L = \sum_{x\in\mathbb{Z}}(\underbrace{\sigma_i}_{acts\ on\ \eta_i,\eta_{i+1}} - I), \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- **Non-closure problem**:

$$\partial_t \mathbb{E}[\eta_t(0)] = \Delta \mathbb{E}[\eta_t(0)] - \mathbb{E}[\eta_t(-1)\eta_t(0)] - \mathbb{E}[\eta_t(0)\eta_t(1)]$$

Hecke algebras

- **Algebra of two-site generators**:
  1. $\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \neq 1, \; i, j \in \mathbb{Z}$

Hecke algebras

- **Algebra of two-site generators**:
  1. $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \neq 1,\ i,j \in \mathbb{Z}$
  2. $\sigma_i^2 = \sigma_i, i \in \mathbb{Z}$ (Hecke relation)

Hecke algebras

- **Algebra of two-site generators**:
  1. $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \neq 1,\ i,j \in \mathbb{Z}$
  2. $\sigma_i^2 = \sigma_i, i \in \mathbb{Z}$ (Hecke relation)
  3. $\sigma_i\sigma_{i+1}\sigma_i - \frac{1}{4}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} - \frac{1}{4}\sigma_{i+1},\ \ i \in \mathbb{Z}$ (Braid relation)

Hecke algebras

- **Algebra of two-site generators**:
  1. $\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \neq 1,\ i,j \in \mathbb{Z}$
  2. $\sigma_i^2 = \sigma_i, i \in \mathbb{Z}$ (Hecke relation)
  3. $\sigma_i \sigma_{i+1} \sigma_i - \frac{1}{4}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} - \frac{1}{4}\sigma_{i+1},\ \ i \in \mathbb{Z}$ (Braid relation)

- **Claim.** Consider a CTMC on $\{0,1\}^{\mathbb{Z}}$ with $L = \sum_{x \in \mathbb{Z}} (\underbrace{\sigma_i}_{acts\ on\ \eta_i, \eta_{i+1}} - I)$,

  $\{\sigma_i\}_{i \in \mathbb{Z}}$ generate Hecke algebra. Assuming reflection symmetry, there are four such chains:
  1. Mixed coalescing-annihilating RW's ;
  2. Annihilating RW's with pairwise immigration;
  3. Coalescencing-branching RW's;
  4. Symmetric exclusion process

Algebraic solution of the non-closure problem.

- $\mathbb{1}_{\eta(x)=0} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbb{1}_{\eta(x)=1} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Algebraic solution of the non-closure problem.

- $\mathbb{1}_{\eta(x)=0} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbb{1}_{\eta(x)=1} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- **Claim.** For all reaction-diffusion models with Hecke symmetry, there is $w \not\parallel v \in \mathbb{R}^2$:
  1. $\sigma w \otimes w = 1$
  2. $\sigma w \otimes v = \sigma v \otimes w = \alpha v \otimes v + \beta w \otimes w$, $\alpha, \beta \in \mathbb{R}$

Algebraic solution of the non-closure problem.

- $\mathbb{1}_{\eta(x)=0} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbb{1}_{\eta(x)=1} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- **Claim.** For all reaction-diffusion models with Hecke symmetry, there is $w \nparallel v \in \mathbb{R}^2$:
  1. $\sigma w \otimes w = 1$
  2. $\sigma w \otimes v = \sigma v \otimes w = \alpha v \otimes v + \beta w \otimes w$, $\alpha, \beta \in \mathbb{R}$

- $\Phi_t(x_1, x_2) := \mathbb{E}_{\eta_0}(\ldots \otimes v_t(x_1) \left(\otimes_{n=x_1}^{x_2-1} w_t(n)\right) \otimes v_t(x_2) \otimes \ldots)$

Algebraic solution of the non-closure problem.

- $\mathbb{1}_{\eta(x)=0} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbb{1}_{\eta(x)=1} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- **Claim.** For all reaction-diffusion models with Hecke symmetry, there is $w \nparallel v \in \mathbb{R}^2$:
  1. $\sigma w \otimes w = 1$
  2. $\sigma w \otimes v = \sigma v \otimes w = \alpha v \otimes v + \beta w \otimes w$, $\alpha, \beta \in \mathbb{R}$
- $\Phi_t(x_1, x_2) := \mathbb{E}_{\eta_0}\left(\ldots \otimes v_t(x_1) \left(\otimes_{n=x_1}^{x_2-1} w_t(n)\right) \otimes v_t(x_2) \otimes \ldots\right)$
- $\Phi_t(x_1, x_1) = 1$

Algebraic solution of the non-closure problem.

- $\mathbb{1}_{\eta(x)=0} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbb{1}_{\eta(x)=1} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- **Claim.** For all reaction-diffusion models with Hecke symmetry, there is $w \nparallel v \in \mathbb{R}^2$:
  1. $\sigma w \otimes w = 1$
  2. $\sigma w \otimes v = \sigma v \otimes w = \alpha v \otimes v + \beta w \otimes w$, $\alpha, \beta \in \mathbb{R}$

- $\Phi_t(x_1, x_2) := \mathbb{E}_{\eta_0}(\ldots \otimes v_t(x_1) \left(\otimes_{n=x_1}^{x_2-1} w_t(n)\right) \otimes v_t(x_2) \otimes \ldots)$

- $\Phi_t(x_1, x_1) = 1$

- $\partial_t \Phi_t(x_1, x_2) =$
  $\alpha\Phi_t(x_1+1, x_2) + \beta\Phi_t(x_1-1, x_2) + \alpha\Phi_t(x_1, x_2-1) + \beta\Phi_t(x_1, x_2+1)$, $x_2 < x_1$

Conclusions

- Hecke symmetry leads to dulaity functions, whose expectations solve linear equations

Conclusions

- Hecke symmetry leads to dulaity functions, whose expectations solve linear equations
- All correlation functions can be determined (Pfaffian point process for RD systems)

Conclusions

- Hecke symmetry leads to dulaity functions, whose expectations solve linear equations
- All correlation functions can be determined (Pfaffian point process for RD systems)
- Related models: real Ginibre matrix model, random Kac polynomials

Conclusions

- Hecke symmetry leads to dulaity functions, whose expectations solve linear equations
- All correlation functions can be determined (Pfaffian point process for RD systems)
- Related models: real Ginibre matrix model, random Kac polynomials
- Link to integrable systems: R-matrices are built from Hecke generators using **Baxterisation**