Spin nematic phase in quantum spin 1 system

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Theoretical and Mathematical Physics
in Cergy Paris, Singapore, and Warwick
9th -10th July 2020

Spin 1 system with SU(2) invariant interactions

Hilbert space $\mathcal{H}_L = \bigotimes_{x \in \{1,\dots,L\}^d} \mathbb{C}^3$

Spin operators $S^{(1)}, S^{(2)}, S^{(3)}$ on $\mathbb{C}^3$ such that $[S^{(1)}, S^{(2)}] = \mathrm{i} S^{(3)}$, etc...

$S_x^{(i)} = S^{(i)} \otimes \mathrm{Id}_{\{1,\dots,L\}^d \setminus \{x\}}$

General SU(2)-invariant hamiltonian:

$$H = - \sum_{\|x-y\|=1} \Big(J_1 \vec{S}_x \cdot \vec{S}_y + J_2 (\vec{S}_x \cdot \vec{S}_y)^2 \Big)$$

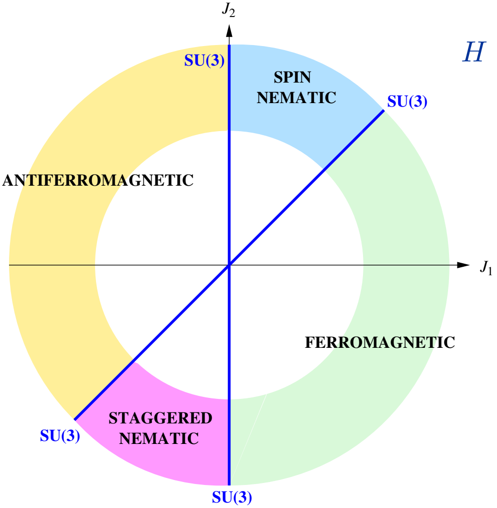
Gibbs state $\langle A \rangle = \dfrac{1}{\mathrm{Tr}\, \mathrm{e}^{-\beta H}} \mathrm{Tr}\, A\, \mathrm{e}^{-\beta H}$

Main question: **low temperature phase diagram** in $d = 3$

Motivation

- Bose-Einstein condensates of integer-spin atoms
- Unconventional magnetic properties of $NiGa_2S_4$ **[Nakatuji et.al. '05]**. Here, spin 1 carrying Ni^{2+} ions reside on weakly coupled triangular lattice layers. Specific heat has T^2 behaviour, akin to linearly dispersing two-dimensional low-energy soft modes. Seems to be related to spin nematic and staggered nematic phases
- Evidence of spin nematic phase in $LiCuVO_4$ **[Zhitomirsky et.al. '10; Svistov et.al. '11]**

Phase diagram for $d \geqslant 2$



$$H = -\sum_{\|x-y\|=1} \big(J_1 \vec{S}_x \cdot \vec{S}_y + J_2 (\vec{S}_x \cdot \vec{S}_y)^2 \big)$$

Phase diagram for $d \geqslant 2$, studied in **[Batista, Ortiz ’04; Tu, Zhang, Xiang ’08; Tóth, Läuchli, Mila, Penc ’12; Fridman, Kosmachev, Klevets ’13]**

Nematic phase

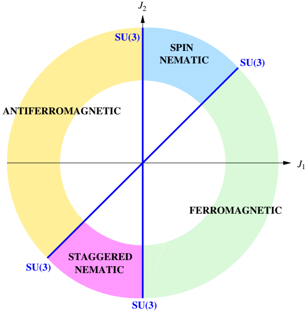
Source: Wikipedia

Motivated by **liquid crystals**. Low temperature phases have *preferred axis* but *no preferred direction*

Spin nematic phases are expected to display

- decay of usual spin correlations: $\langle S_0^{(i)} S_x^{(i)} \rangle \to 0$ as $\|x\| \to \infty$
- long-range order of nematic correlations: $\langle (S_0^{(i)})^2 (S_x^{(i)})^2 \rangle - \langle (S_0^{(i)})^2 \rangle \, \langle (S_x^{(i)})^2 \rangle \not\to 0$ as $\|x\| \to \infty$
- low-temperature Gibbs states are indexed by axes in $\mathbb{S}^2$ (i.e. by the projective sphere $P\mathbb{S}^2$)

Rigorous result from reflection positivity



When $0 \leqslant J_1 < \frac{1}{2} J_2$, the interaction $-J_1 \vec{S}_x \cdot \vec{S}_y - J_2 (\vec{S}_x \cdot \vec{S}_y)^2$ can be shown to be reflection positive

Theorem **[Tanaka², Idokagi '01; U '13]**

Suppose that $d \geqslant 3$, $0 \leqslant J_1 \leqslant \frac{1}{2} J_2$. Then for $\beta > \beta_0$, $\exists\ c > 0$ s.t.

$$\frac{1}{L^d} \sum_{x \in \{1,\dots,L\}^d} \langle (S_0^{(i)})^2 (S_x^{(i)})^2 \rangle - \langle (S_0^{(i)})^2 \rangle^2 > c$$

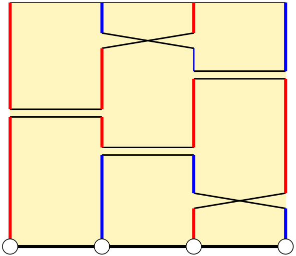
Loop representation for $H = -\sum\big(u\vec S_x\cdot\vec S_y + 1(\vec S_x\cdot\vec S_y)^2\big)$

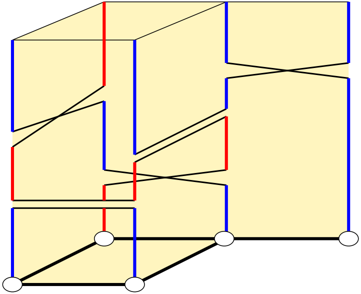
Let ρ be independent Poisson point processes $\times_{\text{edges of }\Lambda}[0,\beta]$, where:

- **crosses** appear with intensity u
- **double bars** appear with intensity $1-u$

$\mathcal{L}(\omega)$: set of loops of the realisation ω

Relevant probability measure: $\boxed{\frac{1}{Z}3^{|\mathcal{L}(\omega)|}\rho(\mathrm{d}\omega)}$



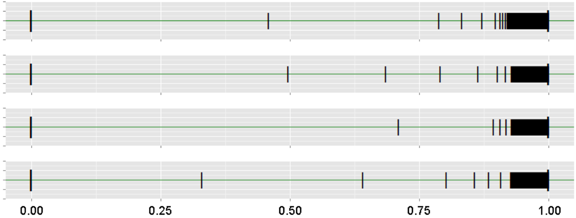


Poisson-Dirichlet conjecture

Given realisation ω, let $\ell_1(\omega), \ell_2(\omega), \dots, \ell_{K(\omega)}(\omega)$ be the lengths of the loops in decreasing order. This is a **random partition** of the interval $[0, 1]$:

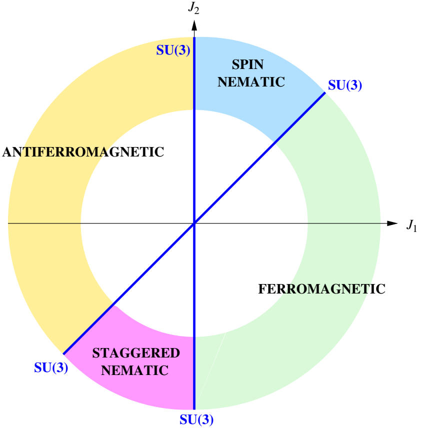
$$\Big(\frac{\ell_1(\omega)}{\beta|\Lambda|}, \frac{\ell_2(\omega)}{\beta|\Lambda|}, \dots, \frac{\ell_{K(\omega)}(\omega)}{\beta|\Lambda|}\Big)$$

Numerical results **[Barp^2, Briol, U '15]**:



Conjecture: The lengths of long loops have joint distribution *Poisson-Dirichlet* $PD(\frac{3}{2})$

"Spin-density Laplace transform"



Assuming that the set of Gibbs states is indexed by $P\mathbb{S}^2$, we have

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \Big\langle \, \mathrm{e}^{\frac{h}{|\Lambda|} \sum_{x \in \Lambda} ((S_x^{(3)})^2 - \frac{2}{3})} \, \Big\rangle =$$

$$= \int_{P\mathbb{S}^2} \mathrm{e}^{h \langle (S_0^3)^2 - \frac{2}{3} \rangle_{\vec{a}}} \, \mathrm{d}\vec{a}$$

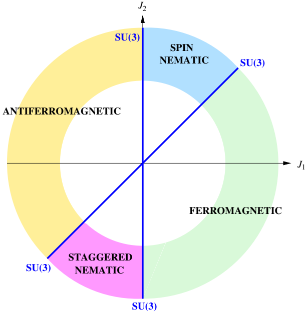
$$= \int_{P\mathbb{S}^2} \mathrm{e}^{h \langle (\vec{a} \cdot \vec{S}_0)^2 - \frac{2}{3} \rangle_{\vec{e}_3}} \, \mathrm{d}\vec{a}$$

$$= \int_{P\mathbb{S}^2} \mathrm{e}^{h \tilde{m} (\frac{3}{2} a_3^2 - \frac{1}{2})} \, \mathrm{d}\vec{a}$$

$$= \mathrm{e}^{-\frac{1}{2} h \tilde{m}} \sum_{k \geqslant 0} \frac{(\frac{3}{2} h \tilde{m})^k}{k! (2k+1)}$$

where $\tilde{m} = \langle (S_0^{(3)})^2 - \frac{2}{3} \rangle_{\vec{e}_3}$

Calculation with PD($\frac{3}{2}$) distribution



$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \Big\langle \mathrm{e}^{\frac{h}{|\Lambda|} \sum_{x\in\Lambda} ((S_x^{(3)})^2 - \frac{2}{3})} \Big\rangle =$$

$$= \mathbb{E}_{\mathrm{PD}(\frac{3}{2})} \Big(\prod_{i \geqslant 1} \big(\tfrac{1}{3}\, \mathrm{e}^{-\frac{2}{3} h m^* \lambda_i} + \tfrac{2}{3}\, \mathrm{e}^{\frac{1}{3} h m^* \lambda_i} \big) \Big)$$

$$= \dots = \mathrm{e}^{-\frac{2}{3} h m^*} \sum_{k \geqslant 0} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+k)} (h m^*)^k$$

where m^* is the density of long loops

But we found $\lim_{\Lambda \nearrow \mathbb{Z}^d} \Big\langle \mathrm{e}^{\frac{h}{|\Lambda|} \sum_{x\in\Lambda} ((S_x^{(3)})^2 - \frac{2}{3})} \Big\rangle = \mathrm{e}^{-\frac{1}{2} h \tilde{m}} \sum_{k \geqslant 0} \frac{(\frac{3}{2} h \tilde{m})^k}{k!(2k+1)}$

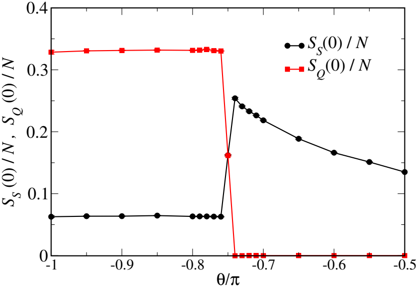
Both red functions are identical provided that

$$\tilde{m} = \langle (S_0^{(3)})^2 - \tfrac{2}{3} \rangle_{\vec{e}_3} = -\tfrac{2}{3} m^*$$

This confirms that the phase is nematic. In addition, states are **planar nematic** (with $S_0^{(3)} \approx 0$) rather than **axial nematic** (with $S_0^{(3)} \approx \pm 1$)

Further evidence of Poisson-Dirichlet

Model on the triangular lattice, in the ground state. The black curve is proportional to the average length of the loop containing 0



[Völl, Wessel '15]

Following the Poisson-Dirichlet conjecture, the ratio of the discontinuity from the right should be $\frac{1/(\frac{3}{2}+1)}{1/(3+1)} = 8/5$

Looking at the figure, the ratio is approximately to $2.6/1.6$. Very close!

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- Fascinating representations of quantum spin systems: random loop models
- Universal behaviour of loop soups in $d \geqslant 3$: Poisson-Dirichlet
- Main physical consequences: study of symmetry breaking
- Many open questions, and further consequences to be discovered!

THANK YOU!