Strange correlations in Symmetry
Protected Topological phases

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Outline

- Symmetry Protected Topological (SPT) phases
- Haldane phase as an example of SPT phase
- Strange correlator to detect gapless edge states
- QMC implementation of strange correlator
- Strange correlator as a probe for topological character of the Haldane phase in $S=1$ Heisenberg spin systems
- Conclusions
Symmetry Protected Topological phases

Simplest category of topological phases - provides an ideal context to study interplay of symmetry, topology, and strong interactions

Protected by symmetries of the ground state

No long range entanglement – only short range entanglement

Vanishing entanglement entropy

No exotic quasiparticle excitations in the bulk

Entanglement spectrum exhibits 2-fold degeneracy

Gapless edge states
The Haldane phase

1D Heisenberg AFM model:

$$\mathcal{H} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

Ground state of $S=1$ Heisenberg AFM chain is the earliest and till date best understood topological phase in interacting quantum many body systems

Ground state: Adiabatically connected to the Affleck-Kennedy-Lieb-Tasaki (AKLT) state

For PBC, ground state is unique:

- product of $s=1/2$ singlets on each bond
- gap to lowest spin excitation – the Haldane gap
- Exponentially decaying spin correlations

No local order parameter; no spontaneously broken (local) symmetry
The Haldane state - string order

There exist topological order that can be measured by the *string order parameter*

In $S^z$ basis, local states of the spins alternate between $S^z=+1$ and $S^z=-1$, with multiple $S^z=0$ sites in between.

No long-range order w.r.t. local observables, i.e. spin-spin correlation decays exponentially with distance.

But a non-local (topological) order exists -- measurable by the “string order parameter”

$$O^\alpha \equiv \lim_{|j-k| \to \infty} \langle S_j^\alpha e^{i\pi \sum_{i=j}^{k-1} S_i^\alpha} S_k^\alpha \rangle, \quad \alpha = x, y, z \quad \text{Den Nijs & Rommelse (1989)}$$

Differentiates the Haldane state from quantum disordered state.
Consider an open chain

There exist one “unpaired / free” s=1/2 (pseudo) moment at each end (edge states)

For a chain of finite length, these end spins interact with each other, but the interaction strength decays exponentially with increasing chain length – $J_{eff} \sim e^{-L/\xi}$

Each edge state (spin) has “approximate” 2-fold symmetry (Kramer’s doublet) - time reversal symmetry

2x2=4 nearly degenerate states whose gap to the ground state vanishes exponentially with increasing chain length – gapless edge states
Strange correlator

- Observable to probe the existence of gapless mode in Symmetry Protected Topological (SPT) states

- Defined as the equal-time correlation of local operator at the imaginary time interface between a topologically trivial state, $\Omega$, and a trial state $\Psi$

  $$ C(r, r') = \frac{\langle \Omega | (S^{x}_{r} S^{x}_{r'} + S^{y}_{r} S^{y}_{r'}) | \Psi \rangle}{\langle \Omega | \Psi \rangle} $$

- If $\Psi$ has topological order, $C(r, r')$ approaches a finite value or decays algebraically as $|r - r'| \rightarrow \infty$

- Very useful observable to probe topological character of arbitrary states.

- Can be understood as follows:
  - The edge states at the spatial interface between topologically trivial and non-trivial states have long-range time correlation.
  - A Lorentz transformation maps the spatial interface to temporal interface and the imaginary time correlation to real space correlation.
Strange correlator

Qualitative understanding

\[ C(r, r') = \frac{\langle \Omega | (S^x_r S^x_{r'}, + S^y_r S^y_{r'}) | \Psi \rangle}{\langle \Omega | \Psi \rangle} \]

- \( |\Psi\rangle \) State generated by evolution under SPT Hamiltonian \( \mathcal{H} \) from \( \tau \rightarrow -\infty \) to \( \tau = 0 \)
- \( |\Omega\rangle \) State generated by evolution under a trivial Hamiltonian \( \mathcal{H}_0 \) from \( \tau \rightarrow +\infty \) to \( \tau = 0 \)
- The edge states at the spatial interface between topologically trivial and non-trivial states have long-range space-time correlation.
- Lorentz transformation maps the spatial interface to temporal interface and the temporal correlation to real space correlation.

You, Bi, Rasmussen, Slagle, Xu, (2014)
 Strange correlator - QMC

- Ideally suited for QMC implementation (provided parent Hamiltonian is sign-problem free)

- QMC studies the evolution in imaginary time – temporal boundary is naturally available.

- Switch the Hamiltonian to one with a known trivial ground state at a given point in the imaginary time.

- Measure equal-time correlations at that imaginary time.

- We use the projective version of the Stochastic Series Expansion QMC algorithm to study the strange correlator in different systems as they are tuned across a topological phase transition. Other QMC algorithms can also be used (e.g., Determinant QMC has been used to study fermionic systems)

Wierschem, P.S. (2014)
Strange correlator - QMC

**Quantitative formulation**

**Projective QMC**: project out the ground state wave function of a Hamiltonian by successive operation on an initial trial state.

Action of multiple operation of a Hamiltonian on a trial state

\[(\mathcal{H} - C_0)^m |\psi_{\text{trial}}\rangle = \sum_{\alpha} c_{\alpha} (E_\alpha - C_0)^m |\alpha\rangle \quad \{|\alpha\rangle\} \text{ complete basis}\]

Projection of the ground state

\[
\left( \frac{\mathcal{H} - C_0}{E_0 - C_0} \right)^m |\psi_{\text{trial}}\rangle = \sum_{\alpha} c_{\alpha} \left( \frac{E_\alpha - C_0}{E_0 - C_0} \right)^m |\alpha\rangle \quad m \rightarrow \infty \quad |\psi\rangle
\]

Evolve a trial state with $\mathcal{H}_0$ and $\mathcal{H}$ from two ends to project out $|\Omega\rangle$ and $|\psi\rangle$ and calculate the correlation at the “middle” of the imaginary time evolution

\[
\mathcal{C}(r, r') = \frac{\langle \psi_{\text{trial}} | (\mathcal{H}_0 - C_0')^m (S^x_r S^x_{r'} + S^y_r S^y_{r'}) (\mathcal{H} - C_0)^m |\psi_{\text{trial}}\rangle}{\langle \psi_{\text{trial}} | (\mathcal{H}_0 - C_0')^m (\mathcal{H} - C_0)^m |\psi_{\text{trial}}\rangle}
\]

Wierschem, P.S. (2014)
Strange correlations in the Haldane state

\[ \mathcal{H} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + D \sum_i (S_i^z)^2 \]

S=1

- **AFM:**
  - Long range magnetic order
  - Spin-spin correlation decays algebraically
  - Finite gap to lowest magnetic excitation

- **Haldane:**
  - No long range magnetic order
  - Spin-spin correlation decays exponentially
  - Finite gap to lowest magnetic excitation

- **QPM:**
  - No long range order
  - Spin-spin correlation decays exponentially
  - Finite gap to lowest magnetic excitation

**String order identifies Haldane phase and can track the phase transition**

**Can the strange correlator do the same?**
**S=1 Heisenberg model with single-ion anisotropy**

\[ \mathcal{H} = \sum_i \vec{S}_i \cdot \vec{S}_j + D \sum_i (S^z_i)^2 \]

The strange correlator correctly identifies the Haldane phase and tracks the phase transition

**Strange correlator**

\[ C(r, r') = \frac{\langle \Omega | (S^x_r S^x_{r'} + S^y_r S^y_{r'}) | \Psi \rangle}{\langle \Omega | \Psi \rangle} \]

**Trivial state**

\[ |\Omega \rangle = \prod_{i=1}^{N} |0 \rangle_i \quad (\text{QPM}) \]

**Projective QMC estimator for strange correlator**

\[ C(r, r') = \frac{\langle \Omega | (S^x_r S^x_{r'} + S^y_r S^y_{r'}) (\mathcal{H} - C)^{2m} | \psi^{tr} \rangle}{\langle \Omega | (\mathcal{H} - C)^{2m} | \psi^{tr} \rangle} \]

**QMC results:**

- **Haldane phase** \( C(r) \rightarrow 0.64 \)
- **Critical region** \( C(r) \sim r^{-\eta} + (L - r)^{-\eta} \)
- **QPM phase** \( C(r) \sim e^{-r/\xi} + e^{-(L-r)/\eta} \)

Wierschem, P.S. (2014)
Strange order parameter

\[ \psi_L = \frac{1}{N} \sum_{r,r'} C(r, r') \]

In the Haldane and QPM phases, the data can be fit with

\[ \psi_L = \psi_\infty + (a - be^{-L/\xi})/L \]

Haldane phase: \( \psi_\infty = 0.64, \xi = 9.58 \)

QPM phase: \( \psi_\infty = 0, \xi = 4.15 \)

In the vicinity of the critical point, the data follows

\[ \psi_L = a/L + b/L^n, \eta = 0.84 \]

Wierschem, P.S. (2014)
S=1 Heisenberg model with single-ion anisotropy

Finite size scaling

critical behaviour is captured by a conformal field theory that maps onto a free Gaussian model

Equal-time GF \( G(r) = \langle S^x_0 S^x_r + S^y_0 S^y_r \rangle \)

Anomalous dimensionality: \( \eta_G = 1/2K \)

Correlation length: critical exponent:
\( \nu = 1/(2 - K) \)

Numerics:
\( D_c = 0.96845(8), \ K = 1.321(1) \)

Strange correlator exhibits finite size scaling behavior with \( \eta_S = 2\eta_G \)

Wierschem, P.S. (2014)
Coupled S=1 Heisenberg chains

2-leg ladder

“free” spins at the ends form inter-chain singlets – ground state is topologically trivial

3-leg ladder

“free” spins at the ends can’t be quenched – ground state is topologically non-trivial

Even-odd effect

how do we correctly identify the nature of the ground state?
Coupled S=1 Heisenberg chains

- String order can’t be used – only defined in 1D
- Strange correlator correctly identifies the topological character of the ground state.
- 2 ways to measure strange correlator – on the same chain ($\Delta x = 0$) and across the chains ($\Delta x = 1$) - both yield same result.

Finite-size scaling of the strange order parameter for 2- and 3-leg ladders
Haldane phase in 2D

\[ \mathcal{H} = J_\parallel \sum_{\langle i,j \rangle_x} \vec{S}_i \cdot \vec{S}_j + J_\perp \sum_{\langle i,j \rangle_y} \vec{S}_i \cdot \vec{S}_j + D \sum_i (S_i^z)^2 \]

Pure Heisenberg model (D=0):

- \( J_\perp \ll J_\parallel \) Weakly coupled chains – the free moments at the chain ends form a 1D \( s=1/2 \) Heisenberg chain with interaction \( J_\perp \) - gapless edge states – Haldane phase survives

- \( J_\perp \to J_\parallel \) Long range 2D AFM (Neel) order sets in, quenching the Haldane phase

Haldane to Neel phase transition marked by (i) closing of Haldane gap, (ii) onset of staggered magnetization, (iii) vanishing of strange order

Spin stiffness

Strange correlator

Correctly identifies SPT character of weakly coupled \( S=1 \) H’berg chains
Haldane phase in 2D

\[ \mathcal{H} = J_{\parallel} \sum_{\langle i,j \rangle_x} \vec{S}_i \cdot \vec{S}_j + J_{\perp} \sum_{\langle i,j \rangle_y} \vec{S}_i \cdot \vec{S}_j + D \sum_i (S^z_i)^2 \]

2D H’berg model with D>0:

- Haldane phase persists at finite, but small, values of D, but is eventually suppressed by sufficiently strong single-ion anisotropy.

- Transition to large-D (QPM) phase marked by vanishing of strange correlation.

Complete ground state phase diagram in 2D

Strange correlation: a key observable in identifying topological phases and phase transitions
Conclusions

- Topological phases are difficult to identify, specially if there in no long range entanglement.

- Gapless edge states are key indicators of the topological nature of a quantum many body state.

- Strange correlator is a powerful probe for detecting gapless edge states.

- Important in the investigation of realistic Hamiltonians with competing interactions.
