Localization, phase and transitions in the three-dimensional extended Lieb lattices

Jie Liu,¹,*  Xiaoyu Mao,¹,†  Jianxin Zhong,¹  and Rudolf A. Römer¹, ², ³

¹School of Physics and Optoelectronics, Xiangtan university, Xiangtan 411105, China
²Department of Physics, University of Warwick, Coventry, CV4 7AL, United Kingdom
³CY Advanced Studies and LPTM (UMR8089 of CNRS), CY Cergy-Paris Université, F-95302 Cergy-Pontoise, France
Anderson localization: Anderson localization (also known as the strong localization) is the absence of diffusion of waves in a disordered medium. Anderson localization is based on interference between multiple scattering paths, leading to localized wave-functions.

Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the “impurity band.” These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions


Background – Flat energy bands

Artificial flat band systems: from lattice models to experiments

Daniel Leykam, Alexei Andreanov and Sergej Flach

Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon, Republic of Korea

DOI: 10.1140/epjb/e2016-70551-2
DOI: 10.1103/PhysRevLett.117.045303
DOI: 10.1103/PhysRevLett.96.126401

(a) 1D pyrochlore lattice, (b) band structure for onsite $\epsilon = 0$.
Model

**Anderson Model:**

\[ H = \sum_i V_i |i\rangle \langle i| - \sum_{i \neq j} t_{ij} |i\rangle \langle j| \]

**Diagonal disorder:**

\[ V_i \in \left[ -\frac{W}{2}, \frac{W}{2} \right] \]

\[ P(V_i) = \frac{1}{W} \left( V_i < \left| \frac{W}{2} \right| \right) \]

\[ P(V_i) = 0 \quad \left( V_i > \left| \frac{W}{2} \right| \right) \]
Method - Transfer Matrix Method

\[ H = \sum_i V_i |i\rangle \langle i| - \sum_{i,j} t_{ij} |i\rangle \langle j| \]

\[ \psi_{n+1} = (E - V_i) \psi_n - \psi_{n-1} \]

\[
\begin{pmatrix}
\psi_{n+1} \\
\psi_n
\end{pmatrix} =
\begin{pmatrix}
E - V_n & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\psi_n \\
\psi_{n-1}
\end{pmatrix}
= T_n
\begin{pmatrix}
\psi_n \\
\psi_{n-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\psi_{L+1} \\
\psi_L
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\begin{pmatrix}
\psi_1 \\
\psi_0
\end{pmatrix}
= \Gamma_L
\begin{pmatrix}
\psi_1 \\
\psi_0
\end{pmatrix}
\]

- \( \Gamma_L \) satisfies Oseledec's theorem, namely that the eigenvalues of the matrix \( (\Gamma_L \Gamma_L^+)^{1/L} \) converges toward \( e^{\pm \gamma} \) as \( L \to \infty \), where \( \gamma \) is known as a Lyapunov exponent.

- The localization length is then determined by the inverse of the minimum Lyapunov exponent

\[ \lambda = \frac{1}{\gamma_{\text{min}}} \]
Method - Transfer Matrix Method

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- The localization length is then determined by the inverse of the minimum Lyapunov exponent

\[ \lambda = \frac{1}{\gamma_{\text{min}}} \]
Method - Finite Size Scaling

According to the renormalization group equation, express \( \Lambda \) as

\[
\Lambda = f\left(\frac{M}{b}, \chi b^\nu, \psi b^\gamma\right)
\]

An appropriate choice of the factor \( b \) leads to,

\[
\Lambda = F\left(\chi M^\nu, \psi M^\gamma\right)
\]

Make a Taylor expansion up to order \( n_1 \)

\[
\Lambda = \sum_{n=0}^{n_1} \psi^n L^{n\gamma} F_n(\chi L^{1/\nu})
\]

Make \( F_n \) Taylor expansion up to order \( n_R \)

\[
F_n(\chi L^{1/\nu}) = \sum_{m=0}^{n_R} \chi^m L^{m/\nu} F_{nm}
\]

Take account of nonlinearity in the scaling variables

\[
\chi(w) = \sum_{n=1}^{m_R} b_n w^n, \quad \psi(w) = \sum_{n=0}^{m_I} c_n w^n
\]

DOI: 10.1103/PhysRevLett.82.382
Result: Dispersion relations

$L3(1)$

$L3(2)$

$L3(3)$

$L3(4)$
Result: Density of states

$\mathcal{L}3(1)$ size: $M^3 = 5^3$

$\mathcal{L}3(2)$ size: $M^3 = 5^3$

$\mathcal{L}3(3)$ size: $M^3 = 4^3$

$\mathcal{L}3(4)$ size: $M^3 = 4^3$
- high precision FSS results
- $W=0$ (analytical)
Critical disorder $W_c = 8.6$
Result: Phase Diagram and FSS for L3(1)
high precision FSS results

$W=0$ (analytical)
Result: Phase Diagram and FSS for L3(2)

Critical disorder \( W_c = 5.96 \)
Critical disorder $W_c = 4.79$
### Conclusion

The critical disorder $W_c$ decreases with an increase.

<table>
<thead>
<tr>
<th>$\Delta M$</th>
<th>$E$</th>
<th>$\delta W$</th>
<th>$n_r$</th>
<th>$m_r$</th>
<th>$L_3(1)$ $W_c$</th>
<th>CI($W_c$)</th>
<th>$\nu$</th>
<th>CI($\nu$)</th>
<th>$p$</th>
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<tbody>
<tr>
<td>16-20</td>
<td>0</td>
<td>8.25-8.9</td>
<td>3</td>
<td>2</td>
<td>8.594</td>
<td>[8.585, 8.604]</td>
<td>1.57</td>
<td>[1.49, 1.65]</td>
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<td>8.25-8.9</td>
<td>3</td>
<td>2</td>
<td>8.598</td>
<td>[8.586, 8.610]</td>
<td>1.55</td>
<td>[1.46, 1.63]</td>
<td>0.08</td>
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<tr>
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<td>8.25-8.9</td>
<td>3</td>
<td>2</td>
<td>8.595</td>
<td>[8.582, 8.607]</td>
<td>1.57</td>
<td>[1.48, 1.66]</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Averages:

<table>
<thead>
<tr>
<th>$\Delta M$</th>
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<th>$\delta E$</th>
<th>$n_r$</th>
<th>$m_r$</th>
<th>$E_c$</th>
<th>CI($E_c$)</th>
<th>$\nu$</th>
<th>CI($\nu$)</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16-20</td>
<td>3</td>
<td>3.725-3.785</td>
<td>3</td>
<td>2</td>
<td>3.748</td>
<td>[3.747, 3.749]</td>
<td>1.75</td>
<td>[1.68, 1.82]</td>
<td>0.88</td>
</tr>
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<td>2</td>
<td>2</td>
<td>3.748</td>
<td>[3.747, 3.749]</td>
<td>1.76</td>
<td>[1.67, 1.84]</td>
<td>0.86</td>
</tr>
<tr>
<td>16-20</td>
<td>3</td>
<td>3.725-3.785</td>
<td>2</td>
<td>2</td>
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<td>[3.747, 3.749]</td>
<td>1.75</td>
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<td>0.86</td>
</tr>
</tbody>
</table>

Averages:

Anderson transition: $W_c = 16.5$

### Conclusion

The critical disorder $W_c$ decreases with an increase.
Result: Critical exponents

\[ \nu \]


K. Slevin and T. Ohtsuki, Physical Review Letters 82, 382 (1999)
The localization properties in all L3(n) lattices show an increased localization with respect to the cubic Anderson lattice and become stronger when n increases.

The overall band width decreases as n increases in all L3(n), and the extremal energy of these bands extends as well towards |E|= 2.

As n increases and the dispersive bands become smaller, the critical properties still retain the universality of the 3D Anderson transition — at least up to n= 3.

The change from 3D dispersive bands with an MIT to a solely 1D system without MIT is not a continuous change, but rather an eventual replacement and shrinking of dispersive bands by a proliferation of flat bands as n grows.

Conclusion
Acknowledgement

➢ Thanks to Professor R. A. Roemer and Professor Jianxin Zhong for their guidance and help!

➢ Thanks to my partner, Xiaoyu Mao for her efforts and help!

➢ Thanks to all the teachers and students present for their questions and listening!

➢ Acknowledge the National Natural Science Foundation of China (Grant No. 11874316), the Program for Changjiang Scholars and Innovative Research Team in University (Grant No. IRT13093), and the Furong Scholar Program of Hunan Provincial Government (R.A.R.) for financial support. This work also received funding by the CY Initiative of Excellence (grant ”Investissements d’Avenir” ANR-16-IDEX-0008) and developed during R.A.R.’s stay at the CY Advanced Studies, whose support is gratefully acknowledged. We thank Warwick’s Scientific Computing Research Technology Platform for computing time and support. UK research data statement: Data accompanying this publication are available from the corresponding authors.