Theoretical and Mathematical Physics in Cergy Paris, Singapore, and Warwick

‘Negative quasiprobability distributions as a measure of nonclassicality’

Dr Bobby Tan
Presidential Postdoctoral Research Fellow
School of Physical & Mathematical Sciences

OUTLINE

- Introducing nonclassicality in light?
- How negative probabilities emerges.
- Applications and interpretations of nonclassicality.

REFERENCES & ACKNOWLEDGMENTS

- More details may be found in:
  - AVS Quantum Sci. 1, 014701 (2019) → Review Paper
  - Phys. Rev. Lett. 119, 190405 (2017)
  - Phys. Rev. Lett. 122, 040503 (2019)
  - Phys. Rev. Lett. 124, 110404 (2020)

- Collaborators: H. Kwon (Imperial College), T. Volkoff (Los Alamos National Laboratory), S. Choi & H. Jeong (Seoul National University).

WHAT IS NONCLASSICAL LIGHT?

- In the classical harmonic oscillator, the system is fully described by a single point in phase space (x, p).

- This is equivalent to knowing precisely the amplitude and relative phase of light in the wave picture.

- In quantum mechanics, this is not possible, due to the uncertainty principle $\Delta x \Delta p \geq \frac{1}{2}$.

- This means that it is not possible to fully represent a classical system in the quantum mechanical framework.

WHAT IS NONCLASSICAL LIGHT?

- Instead, we can ask what is the closest quantum mechanical description of a classical system.

- The closest thing to a point in phase space within quantum mechanics is a minimum uncertainty state:

$$\Delta x \Delta p = \frac{1}{2}$$

- Furthermore, classical mechanics treat position and momentum variables on equal footing, so we should have:

$$\Delta x = \Delta p = \frac{1}{\sqrt{2}}$$

- It can be shown that the only pure states that satisfy this are the set of states called **coherent states**.

COHERENT STATES AS CLASSICAL STATES

- Coherent states are specified using a complex number α:

$$x_0 = \langle \alpha | x | \alpha \rangle = \sqrt{2}\Re(\alpha)$$
$$p_0 = \langle \alpha | p | \alpha \rangle = \sqrt{2}\Im(\alpha)$$

- One may interpret $|\alpha\rangle$ as a state represented by a distribution centered at (x_0, p_0) with uncertainty $\Delta x = \Delta p = \frac{1}{\sqrt{2}}$.

- The output of an ideal laser is a coherent state.

COHERENT STATES AS CLASSICAL STATES

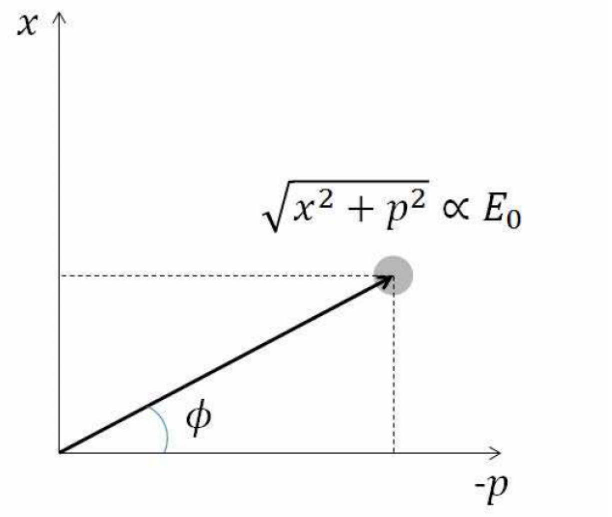


FIG. 1. Interpreting (x, p) phase space coordinates as phasor diagrams. In classical mechanics, a point vector (x, p) represents a plane wave with relative phase ϕ and electric field amplitude E_0 which is proportional to the magnitude $\sqrt{x^2 + p^2}$. In quantum mechanics, it is not possible to represent a state with a single point, due to the uncertainty principle.

COHERENT STATES AS CLASSICAL STATES

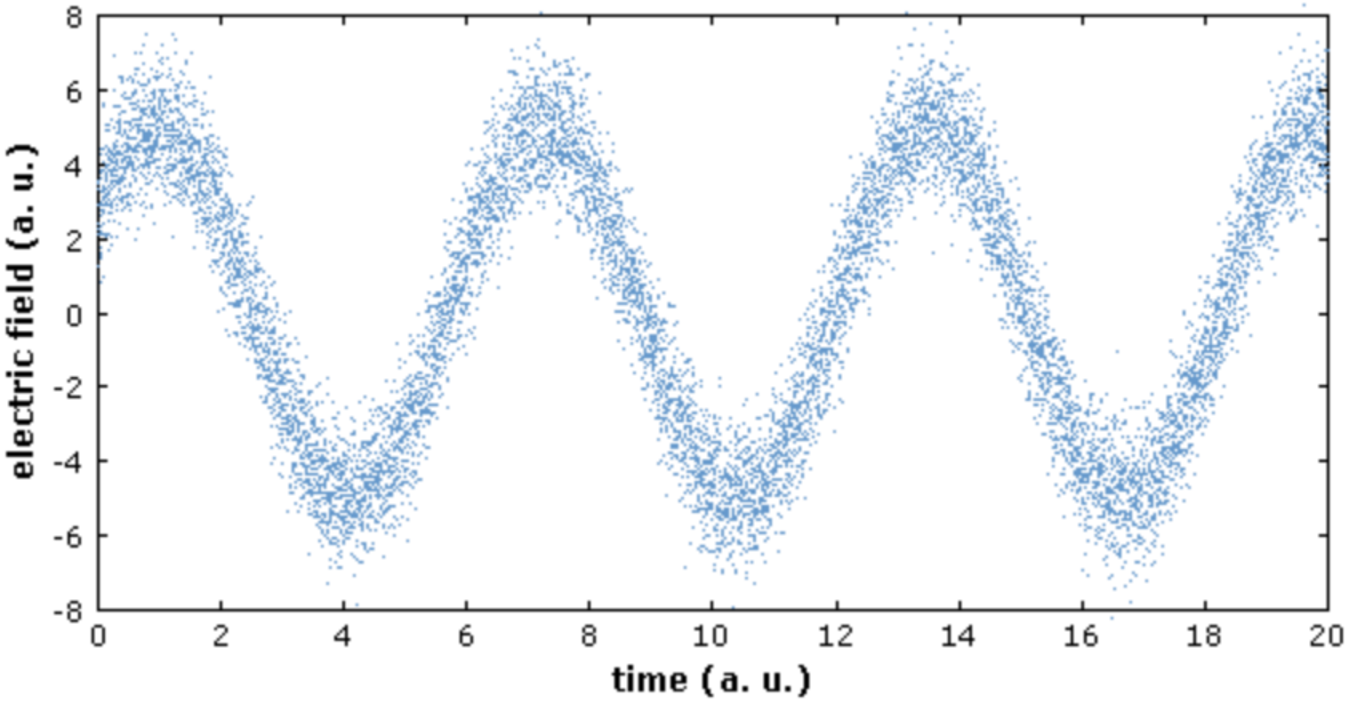
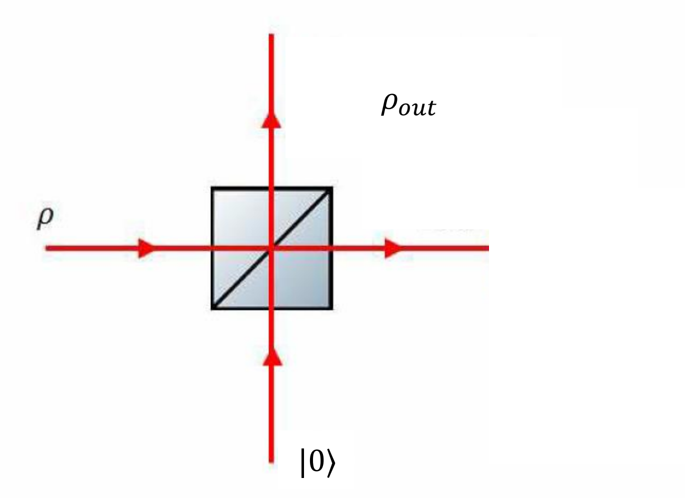


Figure from RP Photonics Encyclopedia

APPLICATIONS OF NONCLASSICAL LIGHT

- Nonclassical states can also be used to generate entanglement for quantum communication purposes.

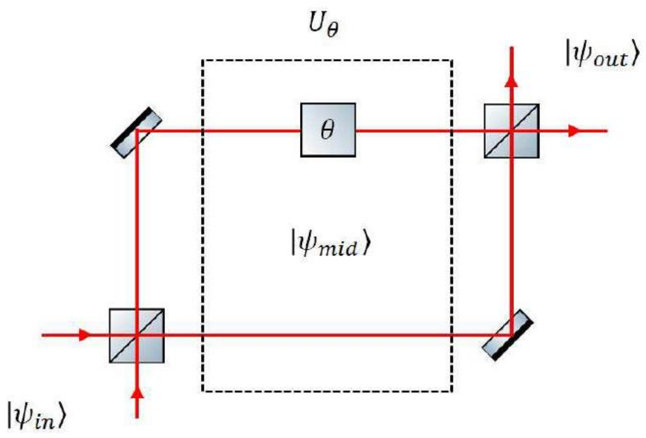


- The output modes ρ_{out} is entangled if and only if the input modes $\rho \otimes |0\rangle\langle 0|$ is nonclassical.

- Entanglement is useful for quantum cryptography or quantum teleportation.

APPLICATIONS OF NONCLASSICAL LIGHT

- They can also be used to overcome shot noise in the Mach-Zender interferometer



- With classical light, the uncertainty of estimating θ scales with $\frac{1}{\sqrt{\langle n \rangle}}$.
- With nonclassical light, the uncertainty of estimating θ scales with $\frac{1}{\langle n \rangle}$.
- Nonclassicality can reduce the resource required to measure a parameter to a certain precision.

APPLICATIONS OF NONCLASSICAL LIGHT

nature physics

Letter | Published: 11 September 2011

A gravitational wave observatory operating beyond the quantum shot-noise limit

The LIGO Scientific Collaboration

Nature Physics **7**, 962–965(2011) | Cite this article

INTRODUCING QUASIPROBABILITY DISTRIBUTION FUNCTIONS

- If you accept the argument that coherent states $|\alpha\rangle$ are the most classical pure quantum states, then a statistical mixture of coherent states must also be classical.

$$\rho_{cl} = \int d^2\alpha P_{cl}(\alpha)|\alpha\rangle\langle\alpha|$$

- Note that here, $P_{cl}(\alpha)$ is a positive probability distribution function over the complex number α.

- In general, if any general density matrix can be written like this, we say that the state of light is classical.

- Otherwise, we say that the state is nonclassical.

INTRODUCING QUASIPROBABILITY DISTRIBUTION FUNCTIONS

- Why are quasiprobability distribution functions relevant in quantum optics?

- It turns out that every quantum state of light can be expressed in the following way:

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$$

- $P(\alpha)$ is called the Glauber-Sudarshan P-function

- One may show that $\int d^2\alpha P(\alpha) = 1$ so it is a quasiprobability distribution.

- If $P(\alpha) \geq 0$ for every α, then ρ is classical.

INTRODUCING QUASIPROBABILITY DISTRIBUTION FUNCTIONS

- The main issue is the P-function suffers from a NAN problem.

- The Fock states $|n\rangle$ has the following P-function:

$$P(\alpha) = L_n\left[-\frac{1}{4}\left(\frac{\partial^2}{\partial\Re(\alpha)^2} + \frac{\partial^2}{\partial\Im(\alpha)^2}\right)\right]\delta(\alpha)$$

- L_n are the Laguerre polynomials of order n.

- It is not easy to analytically interpret such a P-function.

- For the above reason, many physicists prefer to work with other quasiprobability representations.

GENERATING QUASIPROBABILITY DISTRIBUTION FUNCTIONS

- One method of obtaining other quasiprobability representations is apply a "filter".

- Find characteristic function of the P-function, which is its inverse Fourier transform:

$$\chi(\beta) = \mathcal{F}^{-1}P(\beta).$$

- Apply a filtering function $\Phi(\beta)$ to the characteristic function:

$$\chi(\beta) \to \chi(\beta)\Phi(\beta)$$

- Perform the Fourier transform again to obtain quasiprobability representations:

$$\mathcal{F}[\chi(\beta)\Phi(\beta)] = \mathcal{F}[\chi] \star \mathcal{F}[\Phi](\alpha)$$

GENERATING QUASIPROBABILITY DISTRIBUTION FUNCTIONS

- For example, we can choose a Gaussian filter: $\Phi_s(\beta) = e^{-\frac{(1-s)\pi^2}{2}|\beta|^2}$

- This gives rise to s-parametrized quasiprobability functions.

- Different values of s gives rise to different quasiprobabilities.
  a. $s = 1$ is the P-function.
  b. $s = 0$ is the Wigner function.
  c. $s = -1$ is the Q-function.

NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

- When $s \leq 0$, we are guaranteed that $P_s(\alpha)$ is a regular continuous function.

- If any of the quasiprobability distributions $P_s(\alpha)$ displays negativity, then it must have a nonclassical P-function.

- The price to pay is that not all nonclassical states can be detected.

NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

- One way to get around this is to introduce a non-Gaussian filter $\Omega_w(\beta)$.
- By choosing an appropriate non-Gaussian filter, we can ensure that the quasiprobability $P_w(\alpha)$ is a regular function.
- In particular we choose the filter such that if $w \to \infty,\ \Omega_w(\beta) \to 1$.
- This allows us to avoid the NaN problem.

NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

- One may consider the negative volume of the s-parametrized quasiprobabilities to quantify nonclassicality.

$$\mathcal{N}_s(\rho) = \int d^2\alpha P_s^-(\alpha).$$

- However, this definition only makes sense if the quasiprobability outputs a real number.

- By introducing the non-Gaussian filter, this definition now makes sense because we can work with $P_w(\alpha)$ and take $w \to \infty$

NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

- Using this approach, we can also show that the negativity $\mathcal{N}_s$ are linear optical monotones.
- Suppose Φ represents some combination linear optical operations such as mirrors, beam splitters, phase shifters, interferometers etc etc.
- One can show that: $\mathcal{N}_s\big(\Phi(\rho)\big) \leq \mathcal{N}_s(\rho)$.
- This is important because from the quantum information perspective, a nonclassicality measure only makes sense if classical operations does not produce nonclassicality.

NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

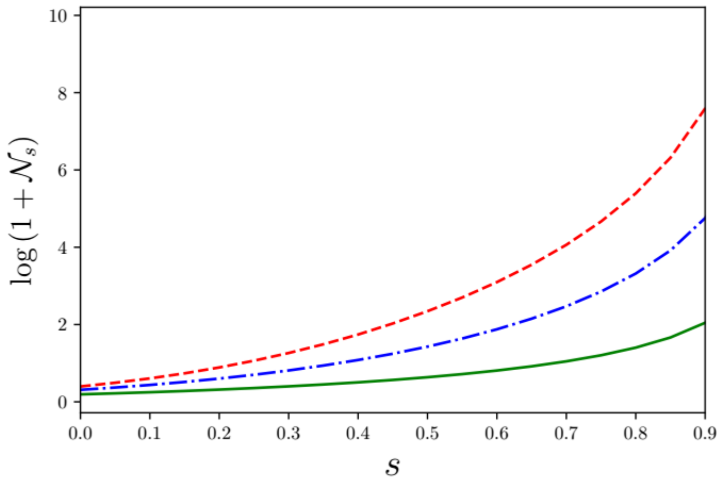


FIG. 2: s-parametrized negativity of Fock state $|n\rangle$ for n =1(solid line), 2(dot-dashed line), and 3(dashed line).

- We can also show that $\mathcal{N}_s$ increases monotonically with s.

- For $s \leq 1, \mathcal{N}_s$ is a lower bound to $\mathcal{N}_{s=1}$

NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

- The negativity of the P-function also has a direct physical interpretation:

- The negativity has a direct operational interpretation as the robustness of nonclassicality:

$$\mathcal{N}(\rho) = \min_{\sigma \in \mathcal{P}} \ \{r \mid r \geq 0, \frac{\rho + r\sigma}{1+r} \in \mathcal{P}\}$$

- The larger the negativity, the more likely nonclassical effects are able to survive a type of classical noise.

CONCLUSION

- Introduced notion of nonclassicality in light.
- Introduced quasiprobability distributions and negativity.
- Discussed how non-Gaussian filters can help to “regularize” quasiprobabilities.
- Discussed the physical significance of the negativity.