‘Negative quasiprobability distributions as a measure of nonclassicality’
OUTLINE

- Introducing nonclassicality in light?
- How negative probabilities emerges.
- Applications and interpretations of nonclassicality.
REFERENCES & ACKNOWLEDGMENTS

More details may be found in:


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WHAT IS NONCLASSICAL LIGHT?

- In the classical harmonic oscillator, the system is fully described by a single point in phase space \((x, p)\).

- This is equivalent to knowing precisely the amplitude and relative phase of light in the wave picture.

- In quantum mechanics, this is not possible, due to the uncertainty principle \(\Delta x \Delta p \geq \frac{1}{2}\).

- This means that it is not possible to fully represent a classical system in the quantum mechanical framework.
WHAT IS NONCLASSICAL LIGHT?

- Instead, we can ask what is the closest quantum mechanical description of a classical system.

- The closest thing to a point in phase space within quantum mechanics is a minimum uncertainty state:

\[ \Delta x \Delta p = \frac{1}{2} \]

- Furthermore, classical mechanics treat position and momentum variables on equal footing, so we should have:

\[ \Delta x = \Delta p = \frac{1}{\sqrt{2}} \]

- It can be shown that the only pure states that satisfy this are the set of states called **coherent states**.
Coherent states are specified using a complex number $\alpha$:

$$x_0 = \langle \alpha | x | \alpha \rangle = \sqrt{2} \Re(\alpha)$$

$$p_0 = \langle \alpha | p | \alpha \rangle = \sqrt{2} \Im(\alpha)$$

One may interpret $|\alpha\rangle$ as a state represented by a distribution centered at $(x_0, p_0)$ with uncertainty $\Delta x = \Delta p = \frac{1}{\sqrt{2}}$.

The output of an ideal laser is a coherent state.
FIG. 1. Interpreting \((x, p)\) phase space coordinates as phasor diagrams. In classical mechanics, a point vector \((x, p)\) represents a plane wave with relative phase \(\phi\) and electric field amplitude \(E_0\) which is proportional to the magnitude \(\sqrt{x^2 + p^2}\). In quantum mechanics, it is not possible to represent a state with a single point, due to the uncertainty principle.
COHERENT STATES AS CLASSICAL STATES

Figure from RP Photonics Encyclopedia
APPLICATIONS OF NONCLASSICAL LIGHT

- Nonclassical states can also be used to generate entanglement for quantum communication purposes.

- The output modes $\rho_{out}$ is entangled if and only if the input modes $\rho \otimes |0\rangle\langle 0|$ is nonclassical.

- Entanglement is useful for quantum cryptography or quantum teleportation.
They can also be used to overcome shot noise in the Mach-Zender interferometer.

- With classical light, the uncertainty of estimating $\theta$ scales with $\frac{1}{\sqrt{\langle n \rangle}}$.
- With nonclassical light, the uncertainty of estimating $\theta$ scales with $\frac{1}{\langle n \rangle}$.
- Nonclassicality can reduce the resource required to measure a parameter to a certain precision.
A gravitational wave observatory operating beyond the quantum shot-noise limit

The LIGO Scientific Collaboration

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INTRODUCING QUASIPROBABILITY DISTRIBUTION FUNCTIONS

- If you accept the argument that coherent states $|\alpha\rangle$ are the most classical pure quantum states, then a statistical mixture of coherent states must also be classical.

\[
\rho_{cl} = \int d^2 \alpha P_{cl}(\alpha) |\alpha\rangle\langle\alpha|
\]

- Note that here, $P_{cl}(\alpha)$ is a positive probability distribution function over the complex number $\alpha$.

- In general, if any general density matrix can be written like this, we say that the state of light is classical.

- Otherwise, we say that the state is nonclassical.
Why are quasiprobability distribution functions relevant in quantum optics?

It turns out that every quantum state of light can be expressed in the following way:

$$\rho = \int d^2 \alpha P(\alpha) |\alpha\rangle \langle \alpha|$$

$P(\alpha)$ is called the Glauber-Sudarshan $P$-function.

One may show that $\int d^2 \alpha P(\alpha) = 1$ so it is a quasiprobability distribution.

If $P(\alpha) \geq 0$ for every $\alpha$, then $\rho$ is classical.
The main issue is the $P$-function suffers from a NAN problem.

The Fock states $|n\rangle$ has the following $P$-function:

$$P(\alpha) = L_n \left[ -\frac{1}{4} \left( \frac{\partial^2}{\partial R(\alpha)^2} + \frac{\partial^2}{\partial S(\alpha)^2} \right) \right] \delta(\alpha)$$

$L_n$ are the Laguerre polynomials of order $n$.

It is not easy to analytically interpret such a $P$-function.

For the above reason, many physicists prefer to work with other quasiprobability representations.
GENERATING QUASIPROBABILITY DISTRIBUTION FUNCTIONS

- One method of obtaining other quasiprobability representations is apply a “filter”.

- Find characteristic function of the $P$-function, which is its inverse Fourier transform:
  \[ \chi(\beta) = \mathcal{F}^{-1}P(\beta). \]

- Apply a filtering function $\Phi(\beta)$ to the characteristic function:
  \[ \chi(\beta) \to \chi(\beta)\Phi(\beta) \]

- Perform the Fourier transform again to obtain quasiprobability representations:
  \[ \mathcal{F}[\chi(\beta)\Phi(\beta)] = \mathcal{F}[\chi] \ast \mathcal{F}[\Phi](\alpha) \]
For example, we can choose a Gaussian filter: $\Phi_s(\beta) = e^{-\frac{(1-s)\pi^2}{2}|\beta|^2}$

This gives rise to $s$-parametrized quasiprobability functions.

Different values of $s$ gives rise to different quasiprobabilities.

- $s = 1$ is the $P$-function.
- $s = 0$ is the Wigner function.
- $s = -1$ is the $Q$-function.
When $s \leq 0$, we are guaranteed that $P_s(\alpha)$ is a regular continuous function.

If any of the quasiprobability distributions $P_s(\alpha)$ displays negativity, then it must have a nonclassical $P$-function.

The price to pay is that not all nonclassical states can be detected.
NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

- One way to get around this is to introduce a non-Gaussian filter $\Omega_w(\beta)$.

- By choosing an appropriate non-Gaussian filter, we can ensure that the quasiprobability $P_w(\alpha)$ is a regular function.

- In particular we choose the filter such that if $w \to \infty$, $\Omega_w(\beta) \to 1$.

- This allows us to avoid the NaN problem.
NEGATIVITY OF QUASIPROBABILITIES AS NONCLASSICALITY MEASURES

- One may consider the negative volume of the $s$-parametrized quasiprobabilities to quantify nonclassicality.

\[ N_s(\rho) = \int d^2 \alpha P_s^-(\alpha). \]

- However, this definition only makes sense if the quasiprobability outputs a real number.

- By introducing the non-Gaussian filter, this definition now makes sense because we can work with $P_w(\alpha)$ and take $w \to \infty$. 
Using this approach, we can also show that the negativity $\mathcal{N}_s$ are linear optical monotones.

Suppose $\Phi$ represents some combination linear optical operations such as mirrors, beam splitters, phase shifters, interferometers etc etc.

One can show that: $\mathcal{N}_s(\Phi(\rho)) \leq \mathcal{N}_s(\rho)$.

This is important because from the quantum information perspective, a nonclassicality measure only makes sense if classical operations does not produce nonclassicality.
We can also show that $\mathcal{N}_s$ increases monotonically with $s$.

For $s \leq 1$, $\mathcal{N}_s$ is a lower bound to $\mathcal{N}_{s=1}$.
The negativity of the $P$-function also has a direct physical interpretation:

The negativity has a direct operational interpretation as the robustness of nonclassicality:

$$\mathcal{N}(\rho) = \min_{\sigma \in \mathcal{P}} \{ r \mid r \geq 0, \frac{\rho + r\sigma}{1+r} \in \mathcal{P} \}$$

The larger the negativity, the more likely nonclassical effects are able to survive a type of classical noise.
CONCLUSION

- Introduced notion of nonclassicality in light.
- Introduced quasiprobability distributions and negativity.
- Discussed how non-Gaussian filters can help to “regularize” quasiprobabilities.
- Discussed the physical significance of the negativity.